7 - Solutions
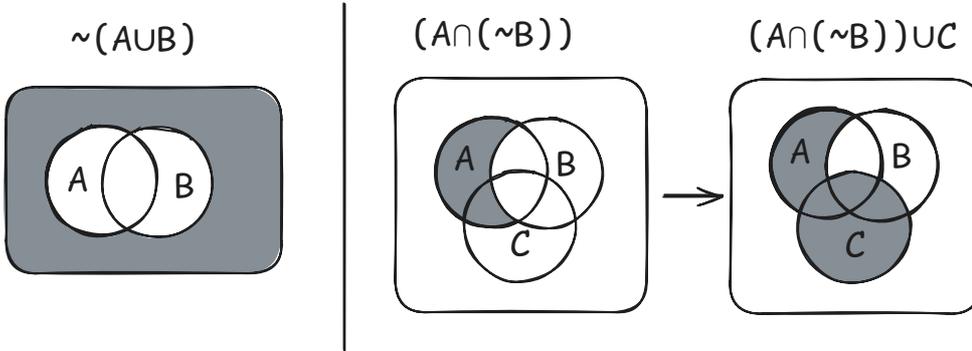
1. Draw the Venn diagram of ~(A∪B). (Review)
2. Draw the Venn diagram of (A∩(~B))∪C. (Review)

Answers:

2. List the elements of the set $\{n \in \mathbb{N} : n \equiv 0 \pmod 3\} \cap [1000, 1010]$.

Answer: The first set consists of multiples of 3. The second set are numbers between 1000 and 1010. Since $1000 \equiv 1 + 0 + 0 + 0 \equiv 1 \pmod 3$ leaves remainder 1 when divided by 3, we know $100 + 2 = 1002$ is divisible by 3, and so is $1002 + 3 = 1005$, and so is $1005 + 3 = 1008$.

Therefore $\{n \in \mathbb{N} : n \equiv 0 \pmod 3\} \cap [1000, 1010] = \{1002, 1005, 1008\}$.

3. A coin is tossed 3 times. How many possible sequences of heads and tails can result?

Answer: The set $\{H, T\}^3$ has cardinality $|\{H, T\}^3| = 2^3 = 8$.

4. A dice is rolled 2 times. How many possible sequences of 1,2,3,4,5,6 can result?

Answer: The set $\{1, 2, 3, 4, 5, 6\}^2$ has cardinality $|\{1, 2, 3, 4, 5, 6\}^2| = 6^2 = 36$.

5. How many 4-digit numeric passcodes use only even digits? Examples: "0240", 8228","2026".

Answer: A 4-digit numeric passcode contains four slots: _,_,_,_. There are five even digits (0, 2, 4, 6, 8) that we can place in each slot, so the answer is $5 \times 5 \times 5 \times 5 = 5^4$.

6. How many 3-digit hexadecimals are there in which the first digit is E or F?

Answer: Such a 3-digit hexadecimal number has three slots: _,_,_. There are two options for the first slot (E,F). There are 16 options for each of the next two slots.

So the answer is $2 \times 16 \times 16$.

7. A standard license plate consists of three letters followed by four digits, like "JRB-4412" and "MMX-8901". How many different standard license plates are possible?

Answer: $26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^2 \times 10^3$.

8. How many subsets of {1, 2, 3, 4, 5, 6, 7} have {2, 3, 5} as a subset?

Answer: Each valid subset of {1, 2, 3, 4, 5, 6, 7} that itself contains {2, 3, 5} can be encoded as an answer to the following sequence of questions:

- Does our subset have 1? Yes or No.
- Does our subset have 2? Yes.
- Does our subset have 3? Yes.
- Does our subset have 4? Yes or No.
- Does our subset have 5? Yes.
- Does our subset have 6? Yes or No.
- Does our subset have 7? Yes or No.

Multiplying these options gives the overall number of options possible:

$$\underset{\substack{\text{Has 1?}\\ \text{Y/N.}}}{2} \times \underset{\substack{\text{Has 2?}\\ \text{Y/N.}}}{1} \times \underset{\substack{\text{Has 3?}\\ \text{Y/N.}}}{1} \times \underset{\substack{\text{Has 4?}\\ \text{Y/N.}}}{2} \times \underset{\substack{\text{Has 5?}\\ \text{Y/N.}}}{1} \times \underset{\substack{\text{Has 6?}\\ \text{Y/N.}}}{2} \times \underset{\substack{\text{Has 7?}\\ \text{Y/N.}}}{2} = \underset{\substack{\text{Overall}\\ \text{options}}}{16}$$

9. How many ways can 1st, 2nd, 3rd places be awarded to 10 runners in a race? (Assume no ties.)

Answer:

$$\underset{\substack{\text{1st place,}\\ \text{pick any of the}\\ \text{10 people}}}{10} \times \underset{\substack{\text{2nd place,}\\ \text{pick any of the 9}\\ \text{remaining people}}}{9} \times \underset{\substack{\text{3rd place,}\\ \text{pick any of the 8}\\ \text{remaining people}}}{8} = \underset{\substack{\text{Overall}\\ \text{options}}}{720}$$

10. How many bitstrings of length 5 start with a 1 or end with a 1?

Answer:

- Let A be the set of bitstrings of length 5 start with a 1. There are 16 of these.
- Let B be the set of bitstrings of length 5 end with a 1. There are 16 of these.

$$A: \quad \underset{\substack{\text{Options:}\\ \text{only 1}}}{1} \times \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} \times \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} \times \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} \times \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} = \underset{\substack{\text{Overall}\\ \text{options}}}{16}$$

$$B: \quad \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} \times \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} \times \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} \times \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} \times \underset{\substack{\text{Options:}\\ \text{only 1}}}{1} = \underset{\substack{\text{Overall}\\ \text{options}}}{16}$$

The intersection $A \cap B$ consist of 5-bit strings that start AND end with a 1. There are 8 of these:

$$A \cap B: \quad \underset{\substack{\text{Options:}\\ \text{only 1}}}{1} \times \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} \times \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} \times \underset{\substack{\text{Options:}\\ \text{0 or 1}}}{2} \times \underset{\substack{\text{Options:}\\ \text{only 1}}}{1} = \underset{\substack{\text{Overall}\\ \text{options}}}{8}$$

Therefore by the principle of inclusion-exclusion (PIE, Lesson 6), the union $A \cup B$ which consists of bitstrings of length 5 start with a 1 or end with a 1, has this many elements:

$$|A \cup B| = |A| + |B| - |A \cap B| = 16 + 16 - 8 = \boxed{24}$$

11. How many 5-digit ternary numbers start with a 2 and end with a 2?

Answer: Ternary numbers can only use $0, 1, 2$ for each of its digits:

$$\underset{\substack{\text{Options:}\\\text{only 2}}}{1} \times \underset{\substack{\text{Options:}\\\text{0, 1, 2}}}{3} \times \underset{\substack{\text{Options:}\\\text{0, 1, 2}}}{3} \times \underset{\substack{\text{Options:}\\\text{0, 1, 2}}}{3} \times \underset{\substack{\text{Options:}\\\text{only 2}}}{1} = \underset{\substack{\text{Overall}\\\text{options}}}{27}$$

$$\underset{\substack{\text{Options:}\\\text{only 2}}}{1} \times \underset{\substack{\text{Options:}\\\text{0, 1, 2}}}{3} \times \underset{\substack{\text{Options:}\\\text{0, 1, 2}}}{3} \times \underset{\substack{\text{Options:}\\\text{0, 1, 2}}}{3} \times \underset{\substack{\text{Options:}\\\text{only 2}}}{1}$$